A STOCHASTIC DIFFERENTIAL MODEL FOR THE EFFECTS OF FAMILY CONFIGURATION ON INTELLIGENCE

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I. Introduction

The model discussed in this study is a refinement and extension of another proposed by the authors in an earlier report (7). As such, it continues the overall process of this research group, that is, refining existing models and developing new ones for the family as a social-psychological system. Implicit in our work is the attempt to improve definition and quantification of the fundamental variables, and the processes interrelating the variables, used in the study of social systems. Also included in this effort is the definition of the variables as random variables.

The variable chosen for this investigation was that of adult intelligence and the development of intelligence in children from birth to age 19 in the context of their family system. The choice was made because of the existence of large and well-conducted studies of the subject (1,8,9), including an early attempt to develop a mathematical model which explained the observed phenomena. In this model, proposed by R.B. Zajonc and G. Markus in 1975 (8), it was speculated that the process proposed to describe the development of individual intellectual capability in a familial context would hold for many other psychological and psychosocial variables. These variables might include other aspects of problem-solving ability in adults, such as creativity, or the development and expression of truly new concepts, affiliative behavior, or the numbers and types of interpersonal relationships formed by the adult, and stability, or the "success" of interpersonal relationships among adults. Other investigators, approaching social systems from different perspectives, have arrived at descriptions of growth processes which are analogous to that proposed by Zajonc and Markus; see (3).

The study of intelligence within a social system also provides the beginnings of a quantitative model for the interaction of biological and sociological variables. This includes the traditional interaction of "nature" and "nurture" to produce individual human capabilities and behaviors. Intelligence, as measured by a specific instrument, is a variable which indicates the biological, or "nature" component, and family configuration, as measured by family size and birth order of the siblings, is a group of variables representing the psychological or "nurture" side.

As mentioned in the earlier report, this type of work had been generally called birth order research, and had been in somewhat of a state of confusion and mild disrepute (3,4) until the publication of the Dutch study of Belmont and Marolla in 1973 (1). Since then, many investigators have agreed that family size and birth order have been definitively shown to be correlated with intelligence, as measured at maturity. Zajonc and Markus proposed a model for a developmental process which could produce the observed outcomes, and which also stressed the importance of spacing between adjacent siblings (in terms of age). The present investigators reported certain refinements in that model (7).

It is the purpose of the present report to indicate how further refinements will improve the model. The refinements include the re-definition of the variates as random variables and the treatment of the interactions among the variables as continuous stochastic processes. The proposed refinements of the model will need to be tested against other sets of data to confirm that it is indeed an improvement in predictive accuracy. But if the improvement can be shown, the model would then have a number of implications for theories of the general function and structure of social systems which will be described in the concluding section of the present work.

II. Formulation of the Model

The authors have previously proposed (7) a deterministic differential equation for the Zajonc and Markus model. This is given by:

$$\dot{M}(t) = \{\ \dot{\ln}\ \alpha(t) - 2k^2t\ \}M(t) + 2k^2t\alpha(t) \tag{2.1}$$

where the dot denotes the derivative with respect to time, t is the age of an individual in years, M(t) is the intellectual level of that individual, and k is an arbitrary rate constant. The function $\alpha(t)$ is a known, continuously differentiable function of time which takes into account the effects of the individual's environment on his development. The initial condition for Equation 2.1 is M(0) = 0. The major assumption in this model concerning the psychological process is that the rate of change of the intellectual development (or the rate of growth of intelligence) is linearly proportional to the intellectual level itself. This formulation, based as it is on the studies of Belmont and Marolla (1) and the work of Zajonc and Markus (9) seems to offer a reasonable and realistic approach.

In our earlier paper, we pointed out that the quantities k and $\alpha(t)$ should, however, more realistically be considered as random variables with certain probability distributions. Treating them as physical constants does not seem to reflect the psychological situation accurately. Thus, some of the uncertainty in measuring intelligence and some of the genetic and environmental differences, both within and between individual families, are expressed in the randomness of these variables in the model. A stochastic version of Equation 2.1 may be expressed by:

$$\dot{M}(t) + A(t)M(t) = Y(t) \qquad (2.2)$$

The solution of Equation 2.2 is a stochastic process, $M(t)$, which gives the intellectual level of an individual at age t.

We will assume that both $A(t)$ and $Y(t)$ are random variables with finite second moments. According to Soong (5), the stochastic process $M(t)$, $0 \leq t \leq T$, is a mean square solution of Equation 2.2 if : (1) $M(t)$ is mean square continuous on the interval $\{0, T\}$, that is, $M(t + h) \overset{ms}{\to} M(t)$ as $h \to 0$ for each $t \geq 0$; (2) $M(0) = 0$ with probability one; and (3) $A(t) \times M(t) + Y(t)$ is the mean square derivative of $M(t)$ on $0, T$. The method of Soong (5, chapter 8) may be utilized to compute the mean square solution of Equation 2.2, which is given by:

$$M(t) = \int_0^t Y(u) \exp\left\{ - \int_u^t A(s)ds \right\}du \qquad (2.3)$$

For a more sophisticated discussion concerning the stochastic structuring of such systems which include the nonlinear cases the reader is referred to the recent monograph of Tsokos and Padgett (6).

In the Zajonc and Markus model, the variable $\alpha(t)$ is essentially flat, almost constant; furthermore, its range is small when compared to the total magnitude of the process $M(t)$. Thus, it would be useful as a first approximation to regard it as constant. In other words, we will take $\alpha(t) = \alpha_0$, where α_0 is a random variable.

We have that:

$$A(t) = 2k^2t \qquad (2.4\text{-i})$$

and:

$$Y(t) = 2k^2t\alpha_0 \qquad (2.4\text{-ii})$$

Therefore, the mean square solution is

$$\begin{aligned} M(t) &= \int_0^t 2k^2\alpha_0 u \exp\left\{ - \int_u^t 2k^2 s ds \right\}du \\ &= \alpha_0 \left\{ 1 - \exp(- k^2t^2) \right\} \qquad (2.5) \end{aligned}$$

which is symbolically identical to the deterministic solution. Knowledge of the probability distributions of the random variables α_0 and k will enable one, through the use of standard techniques such as derived distributions, to calculate the probabalistic behavior of the solution (2.5).

Another logical choice for $\alpha(t)$ would be an exponential function of the form;

$$\alpha(t) = a_0 + a_1 \exp(-c_1t) + a_2 \exp(c_2t), \qquad (2.6)$$

where some or all of the coefficients a_0, a_1, a_2, c_1, and c_2 can be treated as random. The exponential sum is an appropriate form for modelling the behavior described by Zajonc and Markus in (9). They postulated that rather than remain constant, the family process variable $\alpha(t)$ would shift subtly over time, due to the effect of later births, deaths, and other changes in the family's structure. Note that (2.6) is based on the sum of a constant and two other terms, of much lesser magnitude, which reflect deviations from the constant level. Equation 2.6 represents a continuous fit to the step-function, discrete model (7).

Substituting (2.6) in (2.1) and (2.2), we have;

$$A(t) = 2k^2 \frac{-a_1c_1 \exp(-c_1t) + a_2c_2 \exp(c_2t)}{a_0 + a_1 \exp(-c_1t) + a_2 \exp(c_2t)} t \qquad (2.7\text{-i})$$

and;

$$Y(t) = 2k^2t \{a_0 + a_1 \exp(-c_1t) + a_2 \exp(c_2t)\} \qquad (2.7\text{-ii})$$

After putting (2.7) in (2.3) and integrating, the mean square solution is, as before,

$$M(t) = \alpha(t)\{ 1 - \exp(-k^2t^2)\}.$$

This section closes with a comment about the domain $0, T$ of the problem and the level at maturity of $M(t)$. It is easy to see that $M(t)$ is dominated above by $\alpha(t)$. Also, $\alpha(t)$ itself is unbounded for large t in (2.6). The model is defined, however, only for $t \leq T$. The bound T represents the point at which the individual leaves the family setting. This does put a limit on the growth of $M(t)$; it is assumed that for $t > T$, the intellectual level is essentially stable and the period of its rapid increase is over.

III. An Example

This example makes use of the notation and discretized version of the model (2.1) in (7) and (9). Some familiarity on the part of the reader with at least the former reference must be assumed. Because the Dutch data was measured at age 19 on each subject, and was not measured longitudinally through time, it is not possible to estimate the coefficients for $\alpha(t)$ directly. It is possible, on the other hand, to approximate them in the following fashion.

The data consists of 45 scores (transformed) on the Raven Progressive Matrices test, representing the mean score for the ith child in a family with j siblings, $i \leq j$, $i,j \in \{1,2,\ldots,9\}$. These mean values are substituted into the discrete model, which is then solved for 45 coefficients α_{ij}. We form the step function;

$$\alpha'(t) \equiv \alpha_{ij_0} \qquad (3.1)$$

for an ith child, where j_0 is the number of siblings in the family when child i is t years old ($i \leq j_0$, $i,j_0 \in \{1,2,\ldots,9\}$). This representation (3.1) involves the assumption that $\alpha'(t)$ changes only at the birth of each later sibling. The desired coefficients are then estimated by fitting a regression curve to the points where $\alpha'(t)$ jumps, that is, by choosing $\hat{a}_0$, $\hat{a}_1$, $\hat{a}_2$, $\hat{c}_1$, and $\hat{c}_2$ to minimize

$$\sum_{n=1}^{j} \{\alpha'(t_n) - \hat{a}_0 - \hat{a}_1 e^{-\hat{c}_1 t_n} - \hat{a}_2 e^{\hat{c}_2 t_n}\}^2$$

where

t_n = age of child i at the birth of child n.

By choosing k = .1 and assuming the average spacing between successive births to be 2 years, the mean traces in Figure 1 were computed for each sibling in a 9 child family. This graph displays the typical shape of the solution (2.5).

We have thus far only found the mean square solution of the problem, which is identical to the deterministic solution. In an actual problem where more data was available, this would not be of interest. The next step is to study the behavior of the higher order moments of the solution, especially that of the second moment. It is of vital interest to discover the effects of these moments. If, for example, the variance is small relative to the mean, it is not so important to insist upon the stochastic model. The deterministic version could be used. On the contrary, if the variance is large, then the use of the deterministic model could result in a large discrepancy between the predicted behavior and an experimental realization. This might even lead one to discard the basic model as inappropriate.

It is also possible to simulate various outcomes by specifying particular distributions for each of the coefficients, thereby investigating the model under a wide range of conditions. Such broad studies could be of great help in applying this or a similar model to social system variables other than intelligence. In summary, the major objective of this paper has been to discuss, in general terms, how an existing discrete social or psychological model could be treated as a differential equation. This allows the use of well-developed techniques for handling both deterministic and stochastic equations. We have specifically shown how the Zajonc and Markus model might be naturally stochastized. In addition, we proposed a formulation for the process variable $\alpha(t)$ which attempts to reflect the behavior that they postulated for it.

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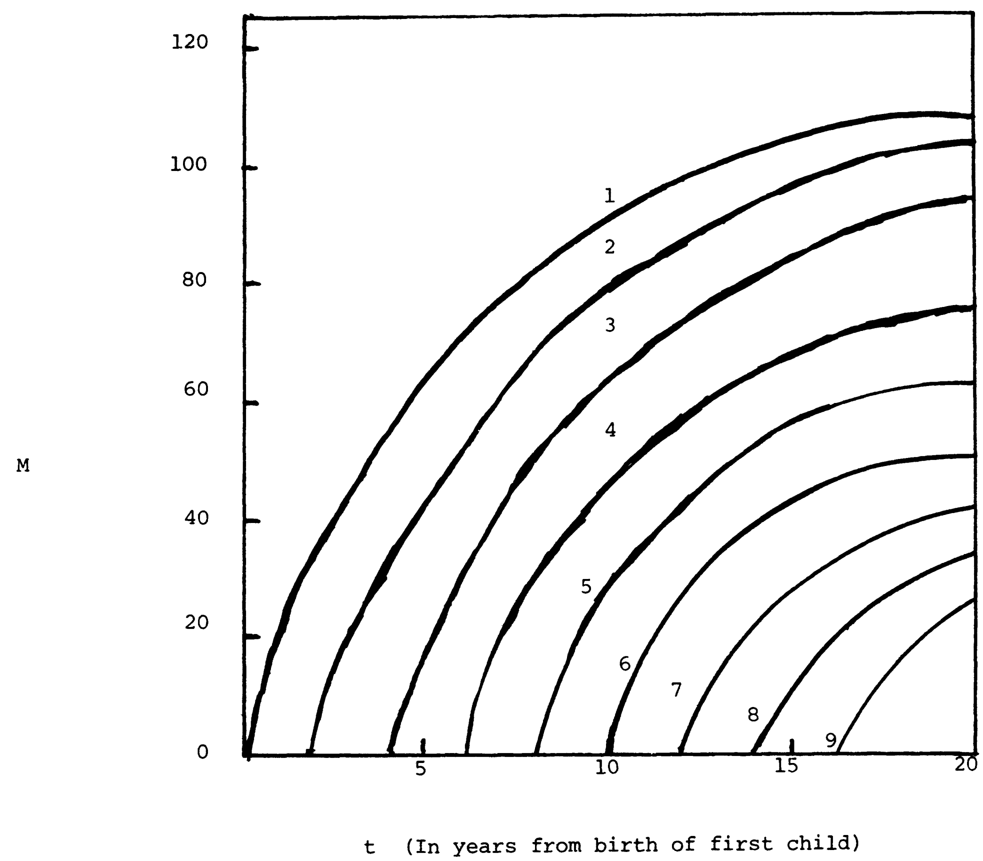
Figure 1: Mean Intellectual Development

For 9 Child Family (Exaggerated)



Δt = Gap between successive births

= 2 years

k = .1